Bayesian Compressive Sensing Framework for Sparse Representations of High-Dimensional Models

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OUTLINE

- Surrogates needed for complex models
- Polynomial Chaos (PC) surrogates do well with uncertain inputs
- Bayesian regression provide results with uncertainty certificate
- Compressive sensing ideas deal with high-dimensionality

Surrogate construction: scope and challenges

Construct surrogate for a complex model $f(\boldsymbol{\lambda})$ to enable

- Global sensitivity analysis
- Optimization
- Forward uncertainty propagation
- Input parameter calibration
- $\cdots$

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion

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Polynomial Chaos surrogate

- Scale the input parameters $\lambda_i \in [a_i, b_i]$

$$\lambda_i = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}\, x_i$$

- Forward function $f(\cdot)$, output u

$$u = f(\boldsymbol{\lambda}(\boldsymbol{x})) \qquad \approx \qquad \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x}) \equiv g(\boldsymbol{x})$$

- Global sensitivity information for free
  - Sobol indices, variance-based decomposition.

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Alternative methods to obtain PC coefficients

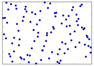
$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Projection $\qquad c_k = \dfrac{\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle}{\langle \Psi_k^2(\boldsymbol{x})\rangle}$

  The integral $\langle u(\boldsymbol{x})\Psi_k(\boldsymbol{x})\rangle = \int u(\boldsymbol{x})\Psi_k(\boldsymbol{x})d\boldsymbol{x}$ can be estimated by

  - Monte-Carlo

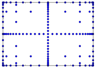
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)$$



many(!) random samples

  - Quadrature

$$\sum_{j=1}^{Q} u(\boldsymbol{x}_j)\Psi_k(\boldsymbol{x}_j)w_j$$



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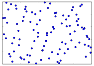

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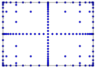
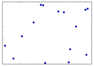


samples at quadrature

- Bayesian regression

$$P(c_k|u(\boldsymbol{x}_j)) \propto P(u(\boldsymbol{x}_j)|c_k)P(c_k)$$



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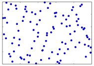

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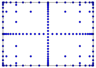

samples at quadrature

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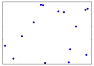
$$\underbrace{P(\boldsymbol{c}|\mathcal{D})}_{\text{Posterior}} \propto \underbrace{P(\mathcal{D}|\boldsymbol{c})}_{\text{Likelihood}}\ \underbrace{P(\boldsymbol{c})}_{\text{Prior}}$$



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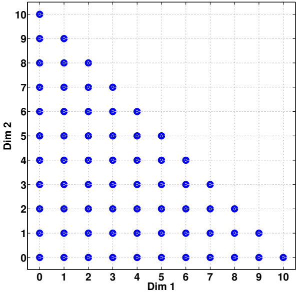
Bayesian inference of PC surrogate: high-d, low-data regime

$$y = u(\boldsymbol{x}) \approx \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

$$\Psi_k(x_1, x_2, ..., x_d) = \psi_{k_1}(x_1)\psi_{k_2}(x_2)\cdots\psi_{k_d}(x_d)$$



- Issues:
  - how to properly choose the basis set?
  - need to work in underdetermined regime $N < K$: fewer data than bases (d.o.f.)
- Discover the underlying low-d structure in the model
  - get help from the machine learning community

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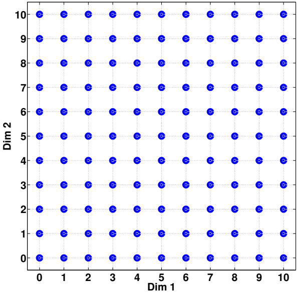

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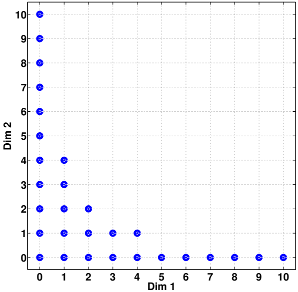

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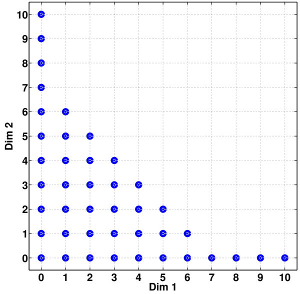

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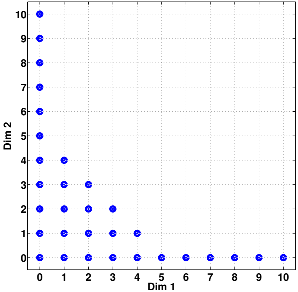

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In a different language....

- N training data points $(\boldsymbol{x}_n, u_n)$ and K basis terms $\Psi_k(\cdot)$
- Projection matrix $\boldsymbol{P}^{N\times K}$ with $\boldsymbol{P}_{nk} = \Psi_k(\boldsymbol{x}_n)$
- Find regression weights $\boldsymbol{c} = (c_0, \dots, c_{K-1})$ so that

$\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c}$ or $u_n \approx \sum_k c_k \Psi_k(\boldsymbol{x}_n)$

- The number of polynomial basis terms grows fast; a p-th order, d-dimensional basis has a total of $K = (p+d)!/(p!d!)$ terms.
- For limited data and large basis set ($N < K$) this is a sparse signal recovery problem $\Rightarrow$ need some regularization/constraints.

- Least-squares *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2\}$
- The ‘sparsest’ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_0\}$
- Compressive sensing *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_1\}$

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Bayesian $\quad$ Likelihood $\quad$ Prior

Bayesian Compressive Sensing (BCS), or Relevance Vector Machine (RVM)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter α can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{P}^T\boldsymbol{u} \qquad \boldsymbol{\Sigma} = \sigma^2(\boldsymbol{P}^T\boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

[Tipping, 2001, Ji *et al.*, 2008; Babacan *et al.*, 2010]

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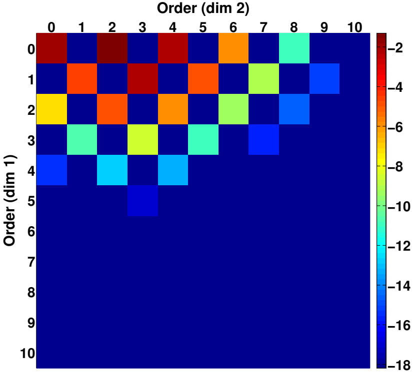
BCS removes unnecessary basis terms

$$f(x,y) = \cos(x + 4y)$$



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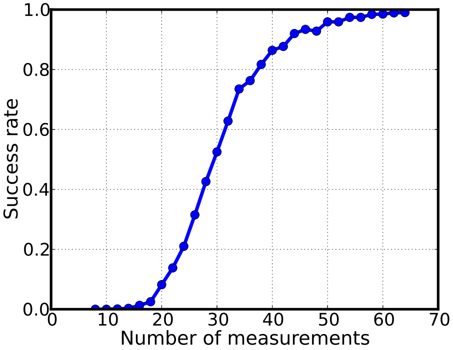
Success rate grows with more data and 'sparser' model

Consider test function

$$f(\boldsymbol{x}) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

where only S coefficients c_k are non-zero. Typical setting is

$$S < N < K$$



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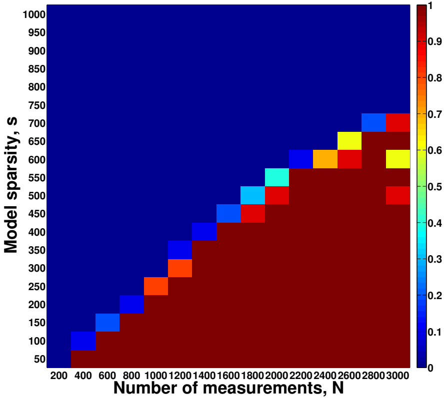
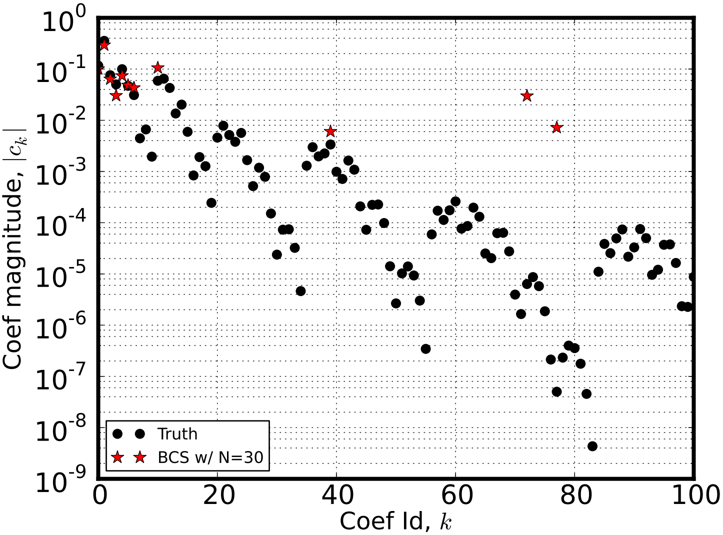


BCS recovers true PC coefficients with increased number of measurements



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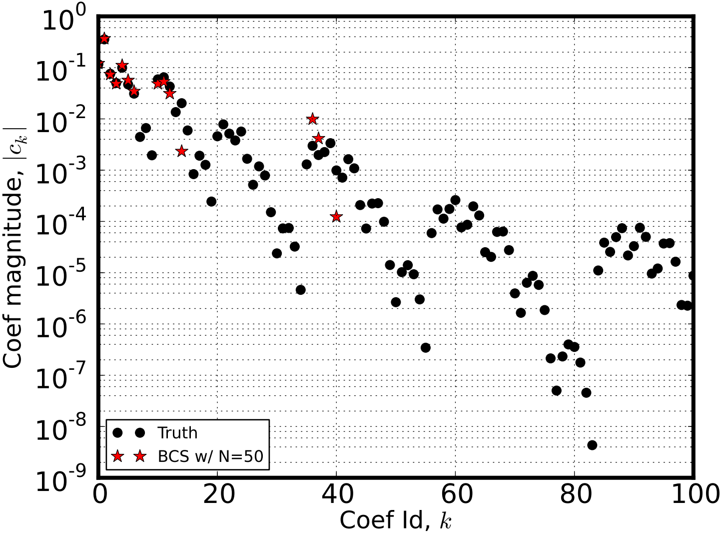

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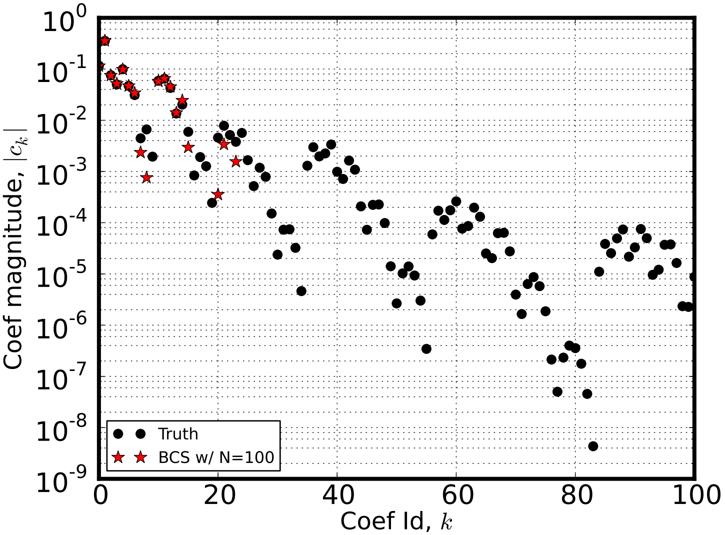

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Weighted Bayesian Compressive Sensing

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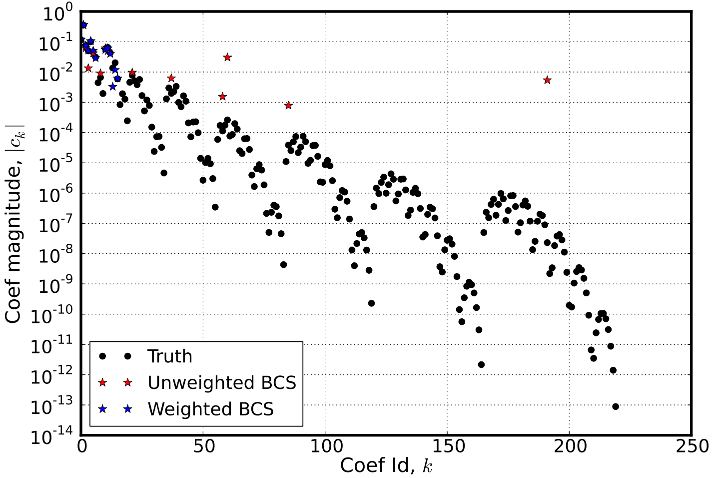
WBCS recovers true coefficients better



$$f(\boldsymbol{x}) = x_0 \cos\left(e + \sum_{i=1}^{9} x_i/i\right)$$

Iteratively reweighting Compressive Sensing [Candes *et al.*, 2007]

Sparsest solution: $min||\boldsymbol{c}||_0$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

Compressive sensing: $min||\boldsymbol{c}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

Weighted compressive sensing: $min||\boldsymbol{Wc}||_1$ such that $\boldsymbol{u} \approx \boldsymbol{Pc}$

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For sparse signals, $\boldsymbol{u} = \boldsymbol{P}\boldsymbol{c}^s$, with $||\boldsymbol{c}_s||_0 = S < K$, ideal weights are

$$\boldsymbol{W} = diag\left(\frac{1}{|c_k^s|}\right) \qquad \text{[i.e., } W_{kk} = +\infty \text{ if } c_k^s = 0\text{]}$$

In practice, the true signal coefficients are not known, so...

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In practice, the true signal coefficients are not known, so...

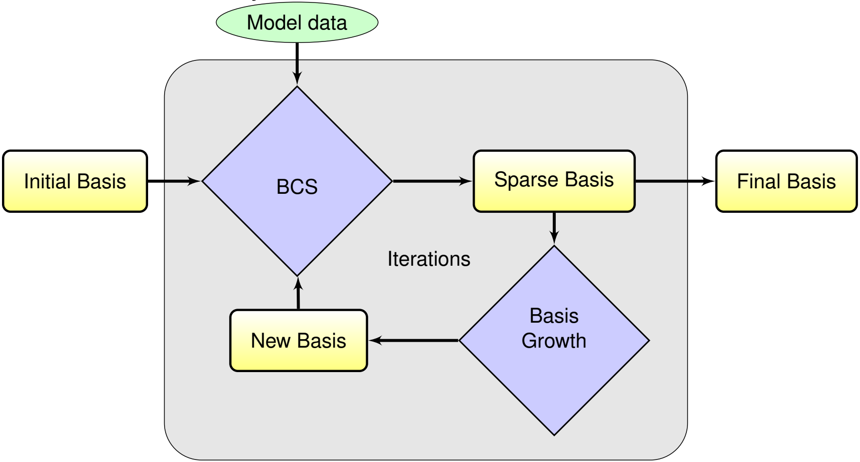
Iterative re-weighting

$$\boldsymbol{W}^{(i+1)} = diag\left(\frac{1}{|c_k^{(i)}| + \epsilon}\right)$$

[$\epsilon \ll 1$ for stability]

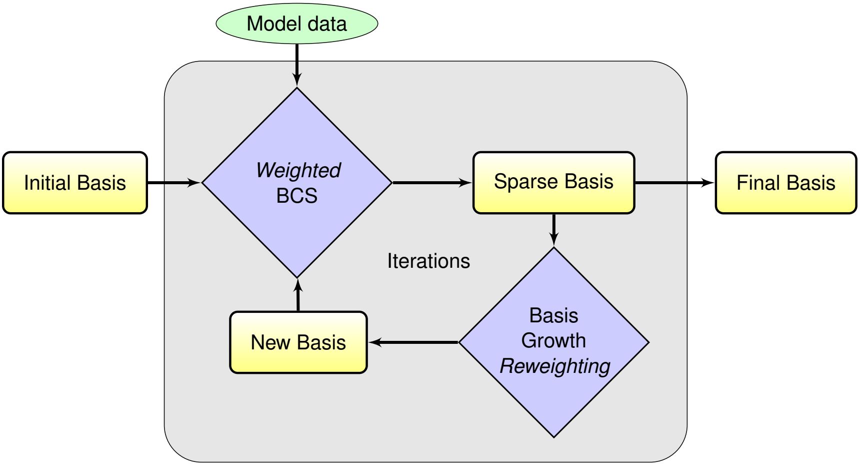
Iterative Bayesian Compressive Sensing (iBCS)

- *Iterative BCS*: We implement an iterative procedure that allows increasing the order for the relevant basis terms while maintaining the dimensionality reduction [Sargsyan *et al.* 2014].



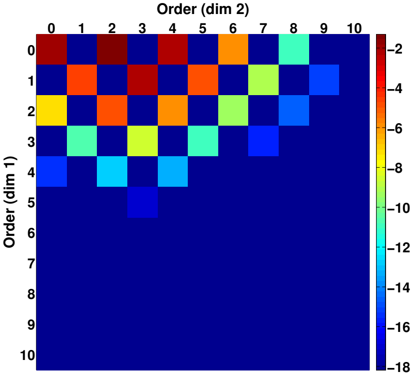
Iterative Bayesian Compressive Sensing (iBCS)

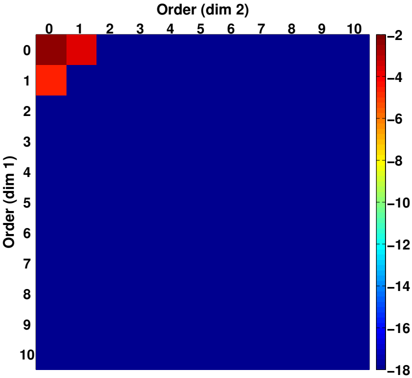
- Combine basis growth and reweighting!



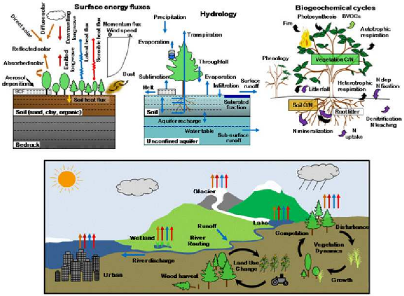
Basis set growth

$$f(x,y) = \cos(x + 4y)$$





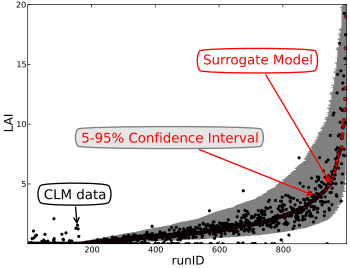
Application of Interest: Community Land Model

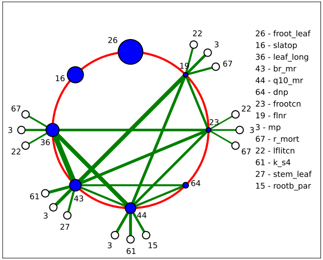


http://www.cesm.ucar.edu/models/clm/

- Nested computational grid hierarchy
- A single-site, 1000-yr simulation takes ~ 10 hrs on 1 CPU
- Involves ~ 70 input parameters; some dependent
- Non-smooth input-output relationship

- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- About 200 polynomial basis terms in the 70-dimensional space
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data





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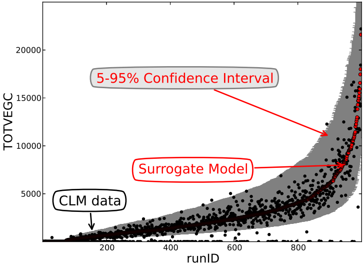

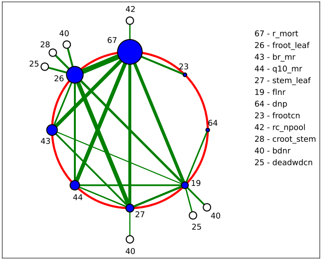

Sparse PC surrogate for the Community Land Model

- Main effect sensitivities : rank input parameters
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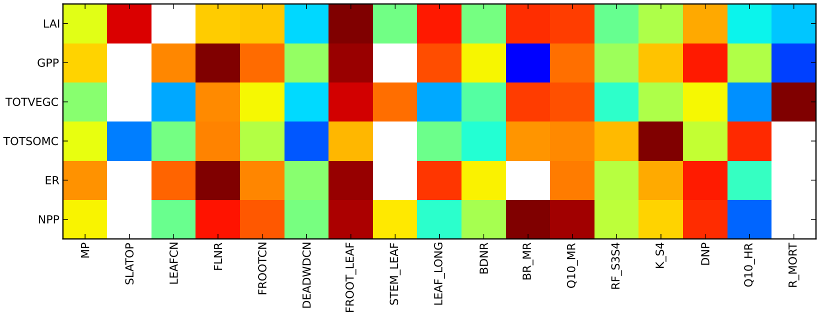

Summary

- Surrogate models are necessary for complex models
  - Replace the full model for both forward and inverse UQ
- Uncertain inputs
  - Polynomial Chaos surrogates well-suited
- Limited training dataset
  - Bayesian methods handle limited information well
- Curse of dimensionality
  - The hope is that not too many dimensions matter
  - Compressive sensing (CS) ideas ported from machine learning
  - We implemented *iteratively* reweighting Bayesian CS algorithm that reduces dimensionality and increases order on-the-fly.

- Future work, open issues
  - Computational design. What is the best sampling strategy?
  - Overfitting still present. Cross-validation techniques help.

Literature

- M. Tipping, "Sparse Bayesian learning and the relevance vector machine", *J Machine Learning Research*, 1, pp. 211-244, 2001.
- S. Ji, Y. Xue and L. Carin, "Bayesian compressive sensing", *IEEE Trans. Signal Proc.*, 56:6, 2008.
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- E. J. Candes, M. Wakin and S. Boyd. "Enhancing sparsity by reweighted ℓ_1 minimization", *J. Fourier Anal. Appl.*, 14 877-905, 2007.
- A. Saltelli, "Making best use of model evaluations to compute sensitivity indices",*Comp Phys Comm*, 145, 2002.

- K. Sargsyan, C. Safta, H. Najm, B. Debusschere, D. Ricciuto and P. Thornton, "Dimensionality reduction for complex models via Bayesian compressive sensing", *Int J for Uncertainty Quantification*, 4(1), pp. 63-93,2014.

Thank You

Random variables represented by Polynomial Chaos

$$X \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta})$$

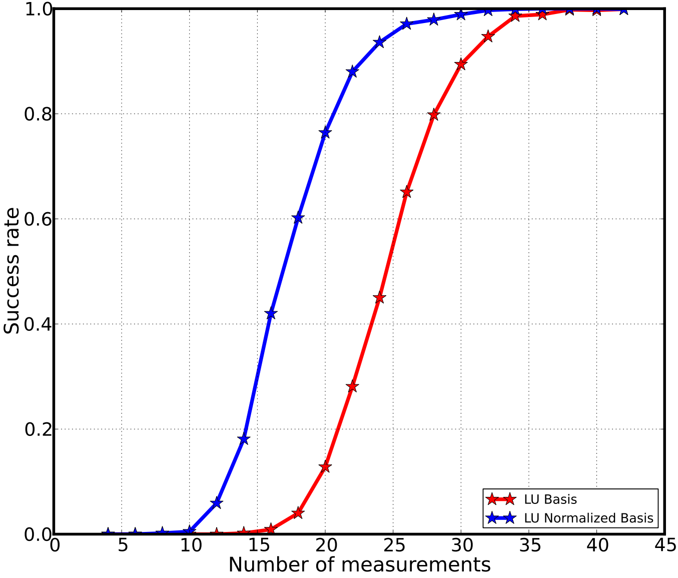
- $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d)$ standard i.i.d. r.v.
  Ψ_k standard polynomials, orthogonal w.r.t. $\pi(\boldsymbol{\eta})$.

$$\Psi_k(\eta_1, \eta_2, \ldots, \eta_d) = \psi_{k_1}(\eta_1)\psi_{k_2}(\eta_2)\cdots\psi_{k_d}(\eta_d)$$

- Typical truncation rule: total-order p, $k_1 + k_2 + \ldots k_d \leq p$.
  Number of terms is $K = \frac{(d+p)!}{d!p!}$.
- Essentially, a parameterization of a r.v. by deterministic spectral modes c_k .
- Most common standard Polynomial-Variable pairs:
  (continuous) Gauss-Hermite, Legendre-Uniform,
  (discrete) Poisson-Charlier.

[Wiener, 1938; Ghanem & Spanos, 1991; Xiu & Karniadakis, 2002; Le Maître & Knio, 2010]

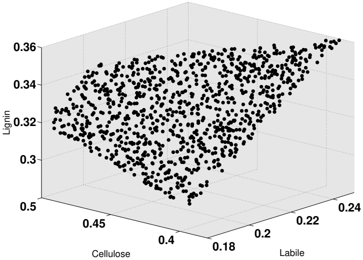
Basis normalization helps the success rate



Input correlations: Rosenblatt transformation

- Rosenblatt transformation maps any (not necessarily independent) set of random variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)$ to uniform i.i.d.'s $\{\eta_i\}_{i=1}^{d}$ [Rosenblatt, 1952].

$$\begin{aligned}
\eta_1 &= F_1(\lambda_1) \\
\eta_2 &= F_{2|1}(\lambda_2|\lambda_1) \\
\eta_3 &= F_{3|2,1}(\lambda_3|\lambda_2, \lambda_1) \\
&\vdots \\
\eta_d &= F_{d|d-1,\ldots,1}(\lambda_d|\lambda_{d-1}, \ldots, \lambda_1)
\end{aligned}$$



- Inverse Rosenblatt transformation $\boldsymbol{\lambda} = R^{-1}(\boldsymbol{\eta})$ ensures a well-defined input PC construction

$$\lambda_i = \sum_{k=0}^{K-1} \lambda_{ik} \Psi_k(\boldsymbol{\eta})$$

- Caveat: the conditional distributions are often hard to evaluate accurately.

Strong discontinuities/nonlinearities challenge global polynomial expansions

- Basis enrichment [Ghosh & Ghanem, 2005]
- Stochastic domain decomposition
  - Wiener-Haar expansions,
    Multiblock expansions,
    Multiwavelets, [Le Maître *et al*, 2004,2007]
  - also known as Multielement PC [Wan & Karniadakis, 2009]
- Smart splitting, discontinuity detection
  [Archibald *et al*, 2009; Chantrasmi, 2011; Sargsyan *et al*, 2011; Jakeman *et al*, 2012]
- Data domain decomposition,
  - Mixture PC expansions [Sargsyan *et al*, 2010]
- Data clustering, classification,
  - Piecewise PC expansions

Piecewise PC expansion with classification

- Cluster the training dataset into non-overlapping subsets $\mathcal{D}_1$ and $\mathcal{D}_2$,
  where the behavior of function is smoother
- Construct global PC expansions $g_i(\boldsymbol{x}) = \sum_k c_{ik}\Psi_k(\boldsymbol{x})$ using each dataset individually $(i = 1, 2)$
- Declare a surrogate

$$g_s(\boldsymbol{x}) = \begin{cases} g_1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_1 \\ g_2(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_2 \end{cases}$$

* Requires a classification step to find out which cluster $\boldsymbol{x}$ belongs to. We applied Random Decision Forests (RDF).

- Caveat: the sensitivity information is harder to obtain.

Illustration of piecewise PC construction

Global 5-th order surrogate fails

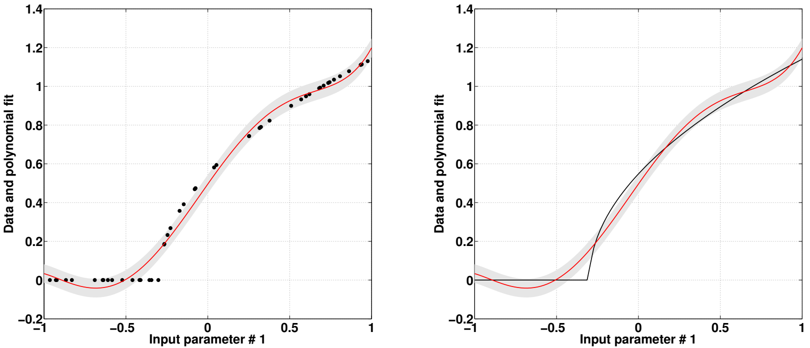
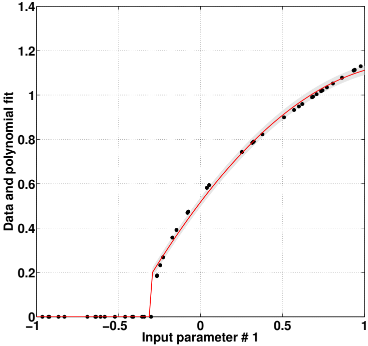


Illustration of piecewise PC construction

Piecewise 2-nd order surrogate



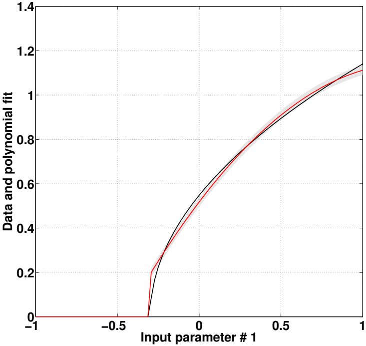


Illustration of piecewise PC construction

Piecewise 5-th order surrogate

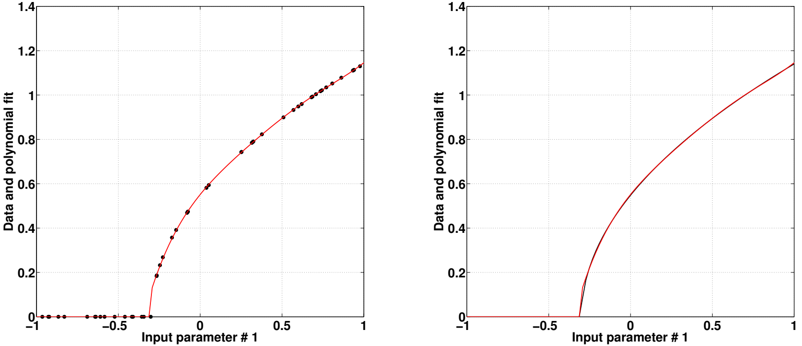


Illustration of piecewise PC construction

Piecewise 5-th order surrogate

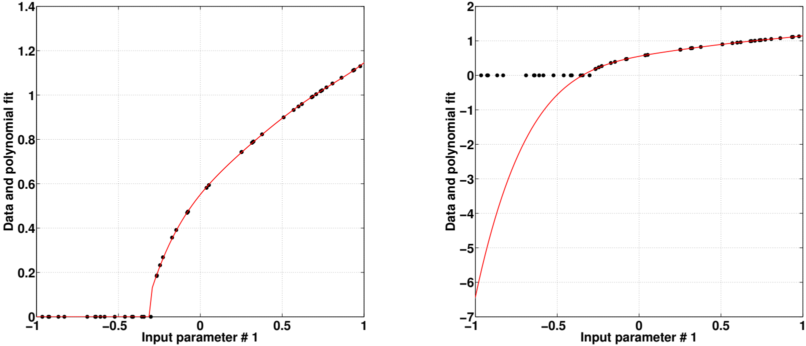
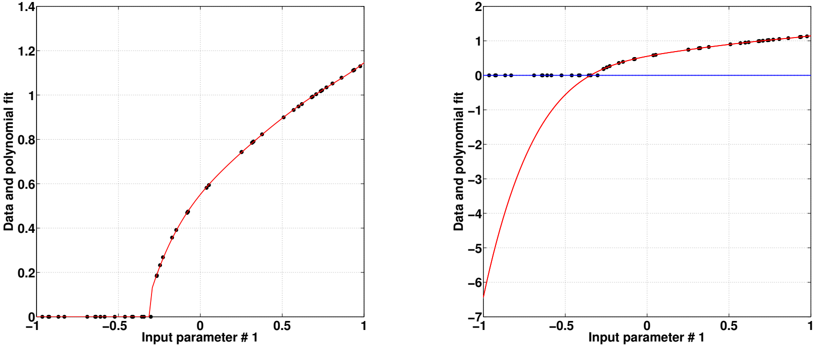


Illustration of piecewise PC construction

Piecewise 5-th order surrogate



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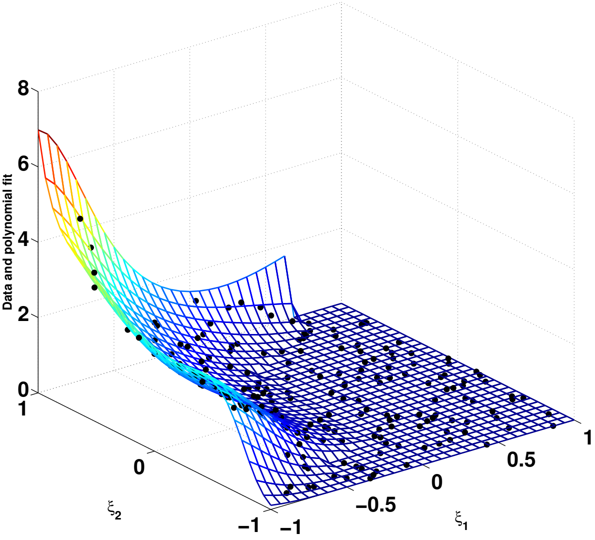

Sensitivity information comes free with PC surrogate,

$$g(x_1, \ldots, x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k \in \mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

$\mathbb{I}_i$ is the set of bases with only x_i involved

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- Joint sensitivity indices

$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i, x_j)]}{Var[g(\boldsymbol{x})]} - S_i - S_j = \frac{\sum_{k \in \mathbb{I}_{ij}} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

$\mathbb{I}_{ij}$ is the set of bases with only x_i and x_j involved

Sensitivity information comes free with PC surrogate,

but not with piecewise PC

$$g(x_1,\ldots,x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k\in\mathbb{I}_i} c_k^2||\Psi_k||^2}{\sum_{k>0} c_k^2||\Psi_k||^2}$$

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- For piecewise PC, need to resort to Monte-Carlo estimation [Saltelli, 2002].