Sparse surrogate model construction via compressive sensing for high-dimensional complex models

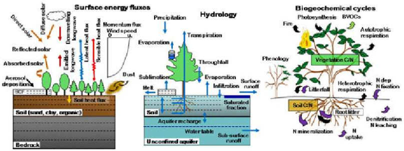
K. Sargsyan, C. Safta, B.Debusschere,H. Najm[1]

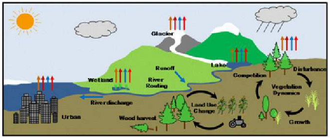
[1]Sandia National Laboratories
Livermore, CA

Sponsored by DOE, Biological and Environmental Research,
under Climate Science for Sustainable Energy Future (CSSEF).

Sandia National Laboratories is a multi-program laboratory operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Application of Interest: Community Land Model





http://www.cesm.ucar.edu/models/clm/

- Nested computational grid hierarchy
- A single-site, 1000-yr simulation takes ~ 10 hrs on 1 CPU
- Involves ~ 70 input parameters; some dependent
- Non-smooth input-output relationship
- CLM simulations courtesy of Dan Ricciuto and Peter Thornton (ORNL)

Challenges we tackle

- Global sensitivity analysis
- Input parameter calibration
- Forward uncertainty propagation

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion
- Input parameter correlations/dependencies
- Strongly non-smooth forward function

Challenges we tackle

- Global sensitivity analysis
- Input parameter calibration
- Forward uncertainty propagation

Construct surrogate for a complex model

- Computationally expensive model simulations, data sparsity
  - Need to build accurate surrogates with as few training runs as possible
- High-dimensional input space
  - Too many samples needed to cover the space
  - Too many terms in the polynomial expansion
- Input parameter correlations/dependencies
- Strongly non-smooth forward function

Random variables represented by Polynomial Chaos

$$X \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta})$$

- $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_d)$ standard i.i.d. r.v.
  Ψ_k standard polynomials, orthogonal w.r.t. $\pi(\boldsymbol{\eta})$.

$$\Psi_k(\eta_1, \eta_2, \ldots, \eta_d) = \psi_{k_1}(\eta_1)\psi_{k_2}(\eta_2)\cdots\psi_{k_d}(\eta_d)$$

- Typical truncation rule: total-order p, $k_1 + k_2 + \ldots k_d \leq p$.
  Number of terms is $K = \frac{(d+p)!}{d!p!}$.
- Essentially, a parameterization of a r.v. by deterministic spectral modes c_k .
- Most common standard Polynomial-Variable pairs:
  (continuous) Gauss-Hermite, Legendre-Uniform,
  (discrete) Poisson-Charlier.

[Wiener, 1938; Ghanem & Spanos, 1991; Xiu & Karniadakis, 2002; Le Maître & Knio, 2010]

Polynomial Chaos surrogate construction

- Build/presume PC for input parameter λ

$$\boldsymbol{\lambda}(\boldsymbol{\eta}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{\eta})$$

with respect to multivariate polynomials of choice.

Polynomial Chaos surrogate construction

- Build/presume PC for input parameter λ

$$\boldsymbol{\lambda}(\boldsymbol{\eta}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{\eta})$$

  with respect to multivariate polynomials of choice.
- e.g., gaussian with known moments, or uniform on an interval,

$$\lambda = \lambda_0 + \lambda_1 \eta$$

Polynomial Chaos surrogate construction

- Build/presume PC for input parameter λ

$$\boldsymbol{\lambda}(\boldsymbol{\eta}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{\eta})$$

with respect to multivariate polynomials of choice.

- If input parameters are uniform $\lambda_i \sim$ Uniform$[a_i, b_i]$, then

$$\lambda_i = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}\, \eta_i.$$

Polynomial Chaos surrogate construction

- Build/presume PC for input parameter λ

$$\boldsymbol{\lambda}(\boldsymbol{\eta}) = \sum_{k=0}^{K-1} \boldsymbol{a}_k \Psi_k(\boldsymbol{\eta})$$

with respect to multivariate polynomials of choice.

- If input parameters are uniform $\lambda_i \sim \text{Uniform}[a_i, b_i]$, then

$$\lambda_i = \frac{a_i + b_i}{2} + \frac{b_i - a_i}{2}\,\eta_i.$$

- Forward function $f(\cdot)$, output u

$$u = f(\boldsymbol{\lambda}(\boldsymbol{\eta})) \qquad\qquad u = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) \equiv g(\boldsymbol{\eta})$$

- Global sensitivity information for free
  - Sobol indices, variance-based decomposition.
- Input dependencies/correlations $\rightarrow$ Rosenblatt transformation

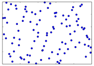
Estimation of PC coefficients

$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) \qquad\qquad c_k = \frac{\langle u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\rangle}{\langle \Psi_k^2(\boldsymbol{\eta})\rangle}$$

The integral $\langle u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\rangle = \int u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}$ can be estimated by

- Monte-Carlo

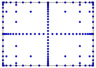
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{\eta}_j)\Psi_k(\boldsymbol{\eta}_j)$$



many samples from $\pi(\boldsymbol{\eta})$

- Quadrature

$$\sum_{j=1}^{Q} u(\boldsymbol{\eta}_j)\Psi_k(\boldsymbol{\eta}_j)w_j$$



samples at quadrature points

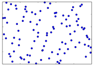
Estimation of PC coefficients

$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) \qquad\qquad c_k = \frac{\langle u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\rangle}{\langle \Psi_k^2(\boldsymbol{\eta})\rangle}$$

The integral $\langle u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\rangle = \int u(\boldsymbol{\eta})\Psi_k(\boldsymbol{\eta})\pi(\boldsymbol{\eta})d\boldsymbol{\eta}$ can be estimated by

- Monte-Carlo

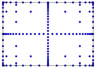
$$\frac{1}{N}\sum_{j=1}^{N} u(\boldsymbol{\eta}_j)\Psi_k(\boldsymbol{\eta}_j)$$



many samples from $\pi(\boldsymbol{\eta})$

- Quadrature

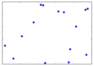
$$\sum_{j=1}^{Q} u(\boldsymbol{\eta}_j)\Psi_k(\boldsymbol{\eta}_j)w_j$$



samples at quadrature points

- *Bayesian inference*

$$P(c_k|u(\boldsymbol{\eta}_j)) \propto P(u(\boldsymbol{\eta}_j)|c_k)P(c_k)$$



any (number of) samples

Bayesian inference of PC surrogate

$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) \equiv g_{\boldsymbol{c}}(\boldsymbol{\eta}) \qquad \overbrace{P(\boldsymbol{c}|\mathcal{D})}^{\text{Posterior}} \propto \overbrace{P(\mathcal{D}|\boldsymbol{c})}^{\text{Likelihood}} \overbrace{P(\boldsymbol{c})}^{\text{Prior}}$$

- Data consists of *training runs*

$$\mathcal{D} \equiv \{(\boldsymbol{\eta}_i, u_i)\}_{i=1}^N$$

- Likelihood with a gaussian noise model with σ^2 fixed or inferred,

$$L(\boldsymbol{c}) = P(\mathcal{D}|\boldsymbol{c}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^N \exp\left(-\frac{(u_i - g_{\boldsymbol{c}}(\boldsymbol{\eta}))^2}{2\sigma^2}\right)$$

- Prior on $\boldsymbol{c}$ is chosen to be conjugate, uniform or Gaussian.
- Posterior is a *multivariate normal*

$$\boldsymbol{c} \quad \in \quad \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Bayesian inference of PC surrogate

$$u \simeq \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) \equiv g_{\boldsymbol{c}}(\boldsymbol{\eta}) \qquad \overbrace{P(\boldsymbol{c}|\mathcal{D})}^{\text{Posterior}} \propto \overbrace{P(\mathcal{D}|\boldsymbol{c})}^{\text{Likelihood}} \overbrace{P(\boldsymbol{c})}^{\text{Prior}}$$

- Data consists of *training runs*

$$\mathcal{D} \equiv \{(\boldsymbol{\eta}_i, u_i)\}_{i=1}^N$$

- Likelihood with a gaussian noise model with σ^2 fixed or inferred,

$$L(\boldsymbol{c}) = P(\mathcal{D}|\boldsymbol{c}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^N \exp\left(-\frac{(u_i - g_{\boldsymbol{c}}(\boldsymbol{\eta}))^2}{2\sigma^2}\right)$$

- Prior on $\boldsymbol{c}$ is chosen to be conjugate, uniform or Gaussian.
- Posterior is a *multivariate normal*

$$\boldsymbol{c} \quad \in \quad \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- The (uncertain) surrogate is *a Gaussian process*

$$\sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{\eta}) = \boldsymbol{\Psi}(\boldsymbol{\eta})^T \boldsymbol{c} \quad \in \quad \mathcal{GP}(\boldsymbol{\Psi}(\boldsymbol{\eta})^T \boldsymbol{\mu}, \boldsymbol{\Psi}(\boldsymbol{\eta}) \boldsymbol{\Sigma} \boldsymbol{\Psi}(\boldsymbol{\eta}')^T)$$

In a different language....

- N training data points $(\boldsymbol{\eta}_n, u_n)$ and K basis terms $\Psi_k(\cdot)$
- Projection matrix $\boldsymbol{P}^{N\times K}$ with $\boldsymbol{P}_{nk} = \Psi_k(\boldsymbol{\eta}_n)$
- Find regression weights $\boldsymbol{c} = (c_0, \dots, c_{K-1})$ so that

$$\boldsymbol{u} \approx \boldsymbol{Pc}$$

- The number of polynomial basis terms grows fast; a p-th order, d-dimensional basis has a total of $K = (p+d)!/(p!d!)$ terms.
- For limited data and large basis set ($N < K$) this is a sparse signal recovery problem $\Rightarrow$ need some regularization/constraints.
- Tikhonov regularization $\quad argmin_{\boldsymbol{c}} \{||\boldsymbol{u} - \boldsymbol{Pc}||_2 + \alpha||\boldsymbol{c}||_2\}$
- Lasso regression $\quad argmin_{\boldsymbol{c}} \{||\boldsymbol{u} - \boldsymbol{Pc}||_2\}$ subject to $||\boldsymbol{c}||_1 \leq \alpha$
- Compressive sensing $\quad argmin_{\boldsymbol{c}} \{||\boldsymbol{u} - \boldsymbol{Pc}||_2 + \alpha||\boldsymbol{c}||_1\}$

In a different language....

- N training data points $(\boldsymbol{\eta}_n, u_n)$ and K basis terms $\Psi_k(\cdot)$
- Projection matrix $\boldsymbol{P}^{N\times K}$ with $\boldsymbol{P}_{nk} = \Psi_k(\boldsymbol{\eta}_n)$
- Find regression weights $\boldsymbol{c} = (c_0, \ldots, c_{K-1})$ so that

$$\boldsymbol{u} \approx \boldsymbol{P}\boldsymbol{c}$$

- The number of polynomial basis terms grows fast; a p-th order, d-dimensional basis has a total of $K = (p+d)!/(p!d!)$ terms.
- For limited data and large basis set ($N < K$) this is a sparse signal recovery problem $\Rightarrow$ need some regularization/constraints.
- Tikhonov regularization $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_2\}$
- Lasso regression $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2\}$ subject to $||\boldsymbol{c}||_1 \leq \alpha$
- Compressive sensing $\quad$ *argmin*$_{\boldsymbol{c}}$ $\{||\boldsymbol{u} - \boldsymbol{P}\boldsymbol{c}||_2 + \alpha||\boldsymbol{c}||_1\}$

Bayesian $\quad$ Likelihood $\quad$ Prior

Bayesian Compressive Sensing (BCS)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad\qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2) p(\sigma_k^2|\alpha) d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter α can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

$$\boldsymbol{\mu} = \sigma^{-2} \boldsymbol{\Sigma} \boldsymbol{P}^T \boldsymbol{u} \qquad\qquad \boldsymbol{\Sigma} = \sigma^2 (\boldsymbol{P}^T \boldsymbol{P} + \mathrm{diag}(\sigma^2/\sigma_k^2))^{-1}$$

[Ji *et al.*, 2008; Babacan *et al.*, 2010]

Bayesian Compressive Sensing (BCS)

- Dimensionality reduction by using hierarchical priors

$$p(c_k|\sigma_k^2) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{c_k^2}{2\sigma_k^2}} \qquad p(\sigma_k^2|\alpha) = \frac{\alpha}{2} e^{-\frac{\alpha\sigma_k^2}{2}}$$

- Effectively, one obtains Laplace *sparsity* prior

$$p(\boldsymbol{c}|\alpha) = \int \prod_{k=0}^{K-1} p(c_k|\sigma_k^2)p(\sigma_k^2|\alpha)d\sigma_k^2 = \prod_{k=0}^{K-1} \frac{\sqrt{\alpha}}{2} e^{-\sqrt{\alpha}|c_k|}$$

- The parameter α can be further modeled hierarchically, or fixed.
- Evidence maximization dictates values for $\sigma_k^2, \alpha, \sigma^2$ and allows exact Bayesian solution

$$\boldsymbol{c} \sim \mathcal{MVN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

with

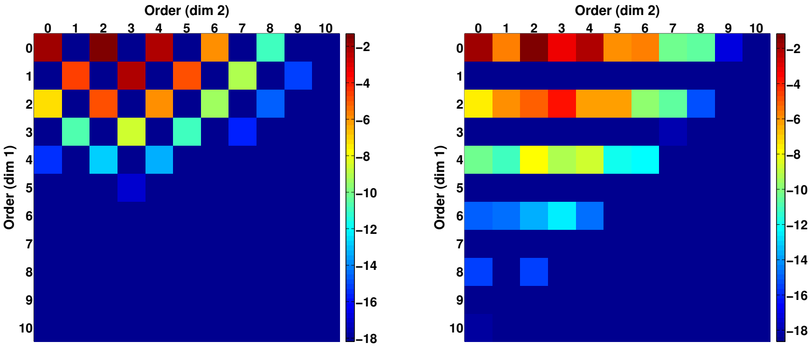
$$\boldsymbol{\mu} = \sigma^{-2}\boldsymbol{\Sigma}\boldsymbol{P}^T\boldsymbol{u} \qquad \boldsymbol{\Sigma} = \sigma^2(\boldsymbol{P}^T\boldsymbol{P} + \text{diag}(\sigma^2/\sigma_k^2))^{-1}$$

- KEY: Some $\sigma_k^2 \to 0$, hence the corresponding basis terms are dropped.

[Ji *et al.*, 2008; Babacan *et al.*, 2010]

BCS removes unnecessary basis terms

$f(x,y) = \cos(x + 4y)$ $\qquad$ $f(x,y) = \cos(x^2 + 4y)$



The square (i,j) represents the (log) spectral coefficient for the basis term $\psi_i(x)\psi_j(y)$.

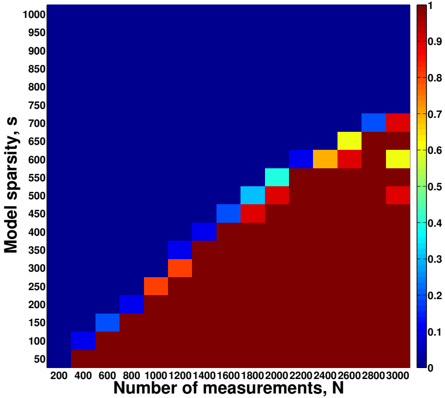
Success rate grows with more data and 'sparser' model

Consider test function

$$f(\boldsymbol{x}) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

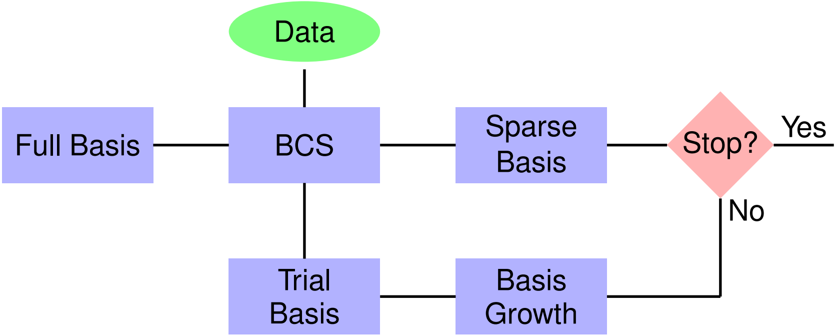
where only S coefficients c_k are non-zero. Typical setting is

$$S < N < K$$



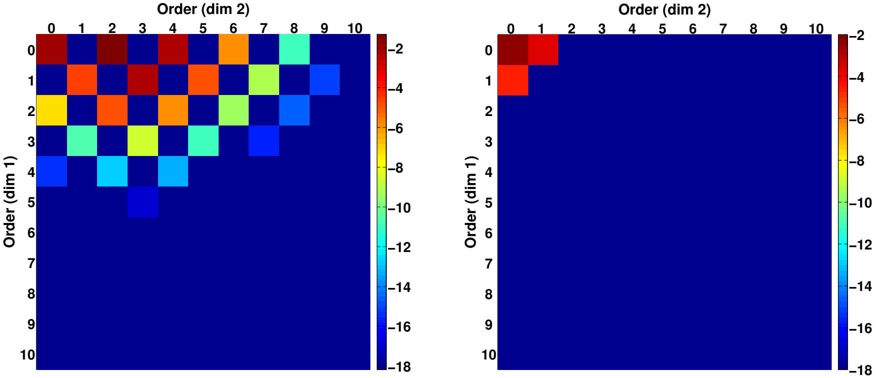
Iterative Bayesian Compressive Sensing (iBCS)

- *Iterative BCS*: We implement an iterative procedure that allows increasing the order for the relevant basis terms while maintaining the dimensionality reduction [Sargsyan *et al.* 2013].



Basis set growth

$$f(x, y) = \cos(x + 4y)$$



Strong discontinuities/nonlinearities challenge global polynomial expansions

- Basis enrichment [Ghosh & Ghanem, 2005]
- Stochastic domain decomposition
  - Wiener-Haar expansions,
    Multiblock expansions,
    Multiwavelets, [Le Maître *et al*, 2004,2007]
  - also known as Multielement PC [Wan & Karniadakis, 2009]
- Smart splitting, discontinuity detection
  [Archibald *et al*, 2009; Chantrasmi, 2011; Sargsyan *et al*, 2011; Jakeman *et al*, 2012]
- Data domain decomposition,
  - Mixture PC expansions [Sargsyan *et al*, 2010]
- Data clustering, classification,
  - Piecewise PC expansions

Piecewise PC expansion with classification

- Cluster the training dataset into non-overlapping subsets $\mathcal{D}_1$ and $\mathcal{D}_2$,
  where the behavior of function is smoother
- Construct global PC expansions $g_i(\boldsymbol{x}) = \sum_k c_{ik}\Psi_k(\boldsymbol{x})$ using each dataset individually ($i = 1, 2$)
- Declare a surrogate

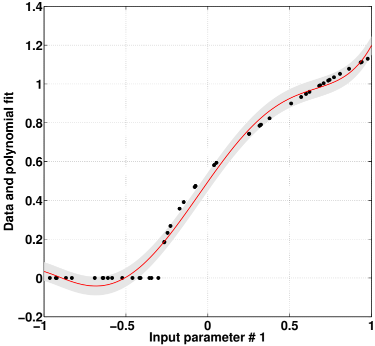
$$g_s(\boldsymbol{x}) = \begin{cases} g_1(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_1 \\ g_2(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in^* \mathcal{D}_2 \end{cases}$$

* Requires a classification step to find out which cluster $\boldsymbol{x}$ belongs to. We applied Random Decision Forests (RDF).

- Caveat: the sensitivity information is harder to obtain.

Illustration of piecewise PC construction

Global 5-th order surrogate fails



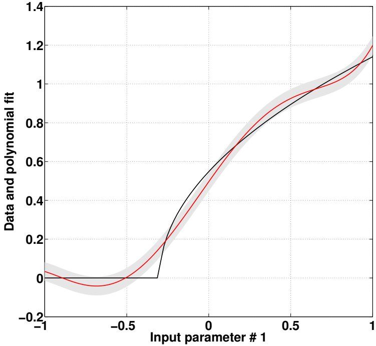
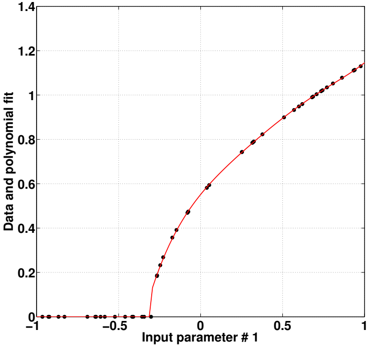


Illustration of piecewise PC construction

Piecewise 5-th order surrogate



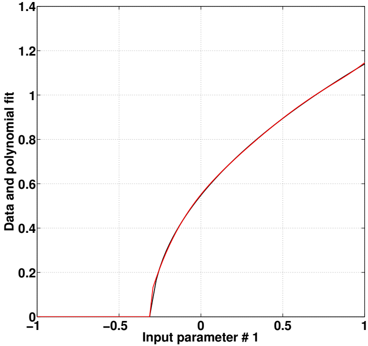
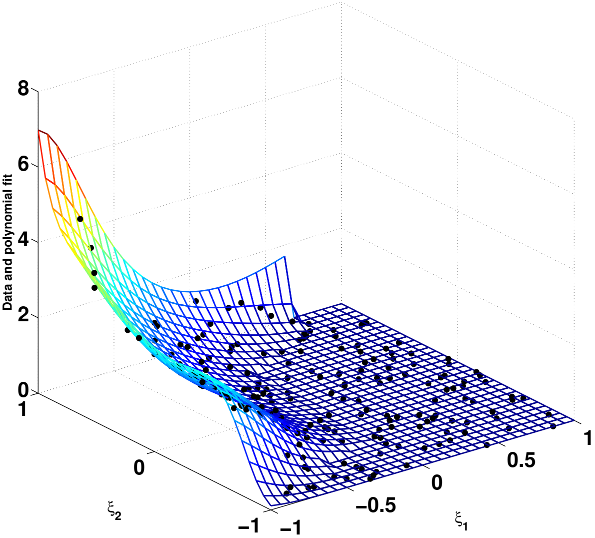
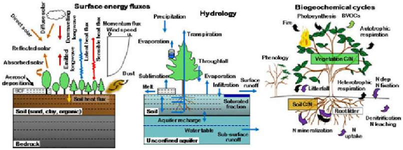
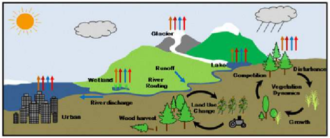


Illustration of piecewise PC construction



Application of Interest: Community Land Model





http://www.cesm.ucar.edu/models/clm/

- Nested computational grid hierarchy
- A single-site, 1000-yr simulation takes ~ 10 hrs on 1 CPU
- Involves ~ 70 input parameters; some dependent
- Non-smooth input-output relationship
- CLM simulations courtesy of Dan Ricciuto and Peter Thornton (ORNL)

Sensitivity information comes free with PC surrogate,

$$g(x_1,\ldots,x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k\in\mathbb{I}_i} c_k^2||\Psi_k||^2}{\sum_{k>0} c_k^2||\Psi_k||^2}$$

$\mathbb{I}_i$ is the set of bases with only x_i involved

- Joint sensitivity indices

$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i,x_j)]}{Var[g(\boldsymbol{x})]} - S_i - S_j = \frac{\sum_{k\in\mathbb{I}_{ij}} c_k^2||\Psi_k||^2}{\sum_{k>0} c_k^2||\Psi_k||^2}$$

$\mathbb{I}_{ij}$ is the set of bases with only x_i and x_j involved

Sensitivity information comes free with PC surrogate,

but not with piecewise PC

$$g(x_1,\ldots,x_d) = \sum_{k=0}^{K-1} c_k \Psi_k(\boldsymbol{x})$$

- Main effect sensitivity indices

$$S_i = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i)]}{Var[g(\boldsymbol{x})]} = \frac{\sum_{k\in\mathbb{I}_i} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

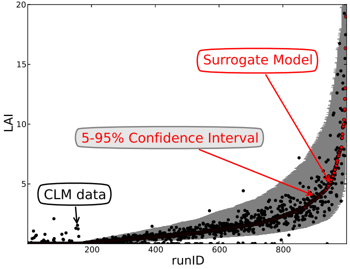
- Joint sensitivity indices

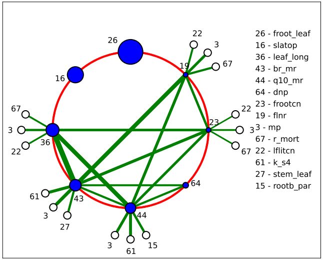
$$S_{ij} = \frac{Var[\mathbb{E}(g(\boldsymbol{x}|x_i,x_j)]}{Var[g(\boldsymbol{x})]} - S_i - S_j = \frac{\sum_{k\in\mathbb{I}_{ij}} c_k^2 ||\Psi_k||^2}{\sum_{k>0} c_k^2 ||\Psi_k||^2}$$

- For piecewise PC, need to resort to Monte-Carlo estimation [Saltelli, 2002].

Sparse PC surrogate for the Community Land Model - 10K ensemble

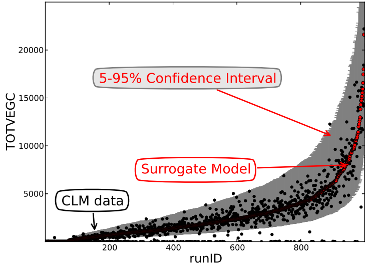
- iBCS leads to about 200 polynomial basis terms in the 70-dimensional space
- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data

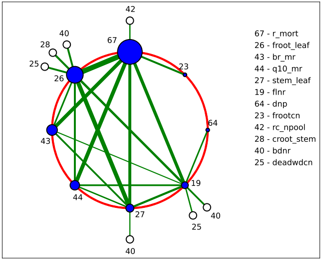




Sparse PC surrogate for the Community Land Model - 10K ensemble

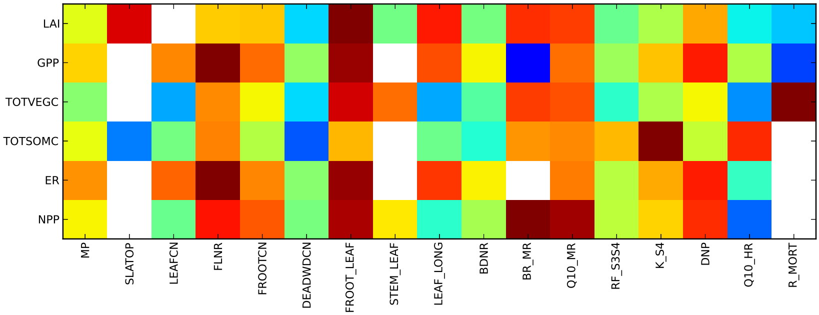
- iBCS leads to about 200 polynomial basis terms in the 70-dimensional space
- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data





Sparse PC surrogate for the Community Land Model - 10K ensemble

- iBCS leads to about 200 polynomial basis terms in the 70-dimensional space
- Main effect sensitivities : rank input parameters
- Joint sensitivities : most influential input couplings
- Sparse PC will further be used for
  - sampling in a reduced space
  - parameter calibration against experimental data



Summary

- Surrogate models are necessary for complex models
  - Replace the full model for both forward and inverse UQ
- Uncertain inputs
  - Polynomial Chaos surrogates well-suited
- Limited training dataset
  - Bayesian methods handle limited information well
- Curse of dimensionality
  - The hope is that not too many dimensions matter
  - Compressive sensing (CS) ideas ported from machine learning
  - We implemented *iterative* Bayesian CS algorithm that reduces dimensionality and increases order on-the-fly.
- Dependent inputs
  - Rosenblatt transformation
- Nonlinear behavior
  - Data clustering and classification-driven piecewise PC
- Applied to CLM
  - Dimensionality reduction, Sensitivity analysis
  - Coming up: lower-dim surrogate and calibration

Literature

- M. Rosenblatt, “Remarks on a multivariate transformation”, *Ann. Math. Statist.*, 23:3, pp. 470-472, 1952.
- S. Ji, Y. Xue and L. Carin, “Bayesian compressive sensing”, *IEEE Trans. Signal Proc.*, 56:6, 2008.
- S. Babacan, R. Molina and A. Katsaggelos, “Bayesian compressive sensing using Laplace priors”, *IEEE Trans. Image Proc.*, 19:1, 2010.
- A. Saltelli, “Making best use of model evaluations to compute sensitivity indices”,*Comp Phys Comm*, 145, 2002.

- K. Sargsyan, C. Safta, H. Najm, B. Debusschere, D. Ricciuto and P. Thornton, “Dimensionality reduction for complex models via Bayesian compressive sensing”, accepted for publication, *Int J for Uncertainty Quantification*, 2013.
- K. Sargsyan, C. Safta, R. Berry, J. Ray, B. Debusschere and H. Najm, “Efficient uncertainty quantification methodologies for high-dimensional climate land models”, Sandia Report, SAND2011-8757, Nov. 2011.

Thank You