Extinction times for birth-death processes: exact results, continuum asymptotics, and the failure of the Fokker-Planck approximation

Submitted to SIAM MMS Special Issue on Multiscale Modeling in Biology

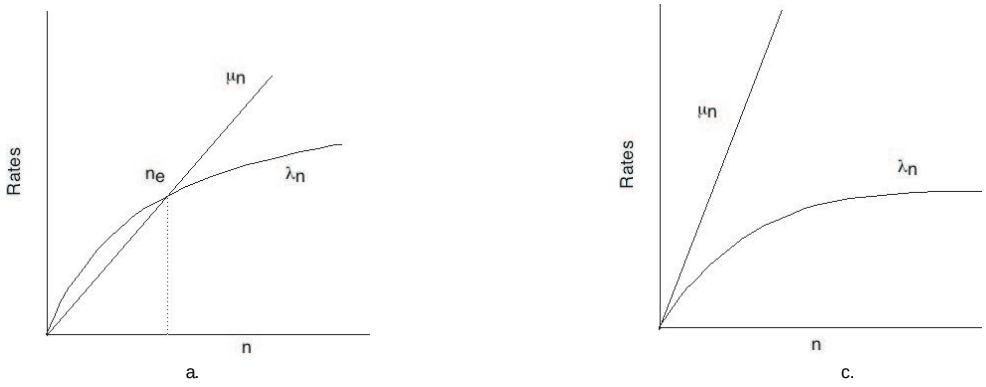
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The setup: a Markov process on states, labelled as $n = 0, 1, ..R \leq \infty$, with following properties:

- birth rate: λ_n is concave down or linear.
- death rate: μ_n is concave up or linear.
- absorbing state: $\lambda_0 = 0$.
- λ_n, $\mu_n \sim n$ for small n.
- dependence on the one control parameter, which defines a threshold.
- with "population scale" N, with $N \to \infty$ as a deterministic(continuum) limit.



Typical model, Susceptible-Infected-Susceptible (SIS):

$$\lambda_n = \Lambda n(1 - n/N)$$

$$\mu_n = n$$

where $N = R$, and the parameter Λ is the infection rate.

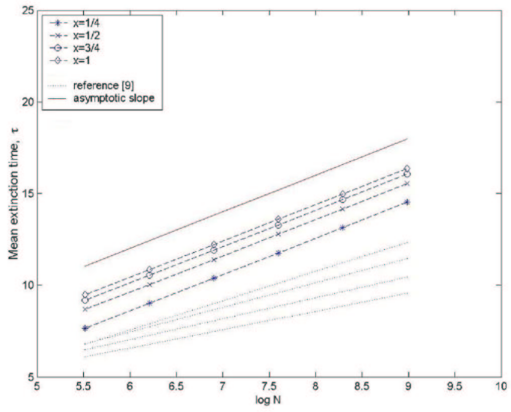
The mean extinction time, starting from the n-th state:

$$\tau_n = \sum_{m=1}^{n} \left(\frac{1}{\mu_m} + \prod_{i=1}^{m-1} \frac{\mu_i}{\lambda_i} \sum_{j=m+1}^{R} \frac{1}{\mu_j} \prod_{k=1}^{j-1} \frac{\lambda_k}{\mu_k}\right)$$

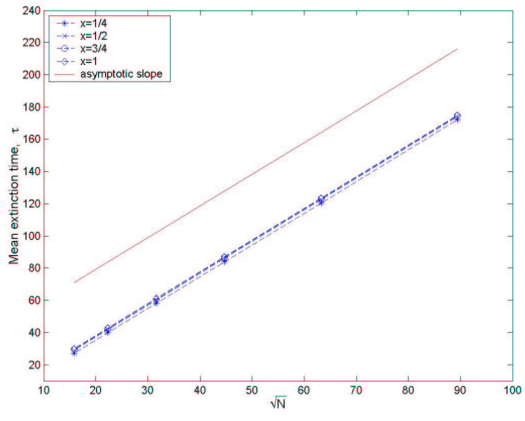
Continuum asymptotics: the leading order of the mean extinction time for large N,

- Subthreshold case,$(\Lambda < 1)$ $\tau_n \propto \log N$
- Threshold case,$(\Lambda = 1)$ $\tau_n \propto \sqrt{N}$
- Superthreshold case,$(\Lambda > 1)$ $\tau_n \propto \frac{e^{N\text{Const}}}{\sqrt{N}}$

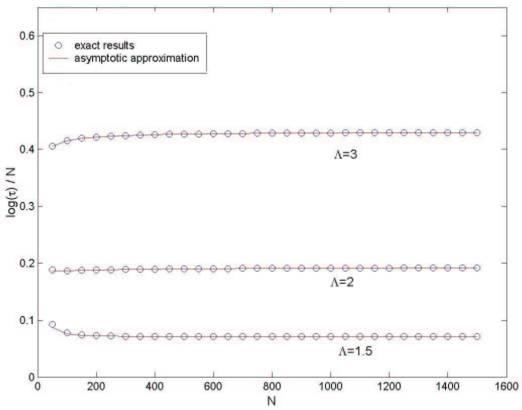
Subthreshold case:



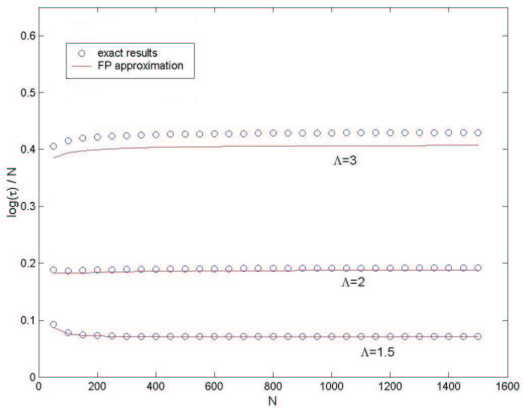
Threshold case:



Superthreshold case:



Superhreshold case(FP approximation):



The failure of the Fokker-Planck approximation:
Approximate the difference equation by a differential one and take the solution of that as an asymptotic limit.
Shown to be correct only in some special cases.